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A Quantiled Study of Dynamic Generalized Non-Additive Entropy Measure for Record Value

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Abstract

Nonnegative entropy measures (Harda and Charvat 1988) for record statistics are essential for numerous applications. But this measure is only relevant for a new or unused function. If a function is used for some unit of time, then what about residual life reliability or failure? In this research paper, we study the quantile approach of Residual Non- additive Entropy for record statistics and some specific distributions. Also, we study the proposed measure in the hazard rate function. At the end graphical presentation makes more deeply and easy to understand the specific features of the measure.

Keywords: Probability distribution; Random variable, uncertainty; Residual Entropy; Quantile function; Record values.

1 Introduction

In 1948 Shannon answered a universal question. If everything in this universe is ordered or disordered then, how much? during his published research paper entitled "Mathematical theory of Communication". In this research paper, a measure of uncertainty named Shannon's entropy was studied. This measure defines the mathematical order of a thing. More especially if X is a nonnegative discrete random variable and p_i is the corresponding probability mass function then the uncertainty measure proposed by Shannon et al. (1948) is given as

$$H(P) = -\sum_{i=1}^{n} p_i \ln p_i. \tag{1.1}$$

If X is a non-negative random variable, $f(x)$ and $F(x)$ are probability density function and probability distribution function then analogous to (1.1) entropy measure for continuous random variable is given as

$$H(X) = -\int_0^\infty f(x) \ln f(x) dx, \tag{1.2}$$

This measure is additive in nature and has applications in many fields like Economics, environmental sciences, coding, decoding, and prediction of data we can refer to Shannon (1948). One parameter additive generalized entropy studied by Renyi in 1961, given as

$$H_{\gamma}(x) = (1-\gamma)^{-1} \ln \left(\int_0^{\infty} f^{\gamma}(x) \, dx \right) \tag{1.3}$$

where $f(x)$ is *pdf* and γ is parameter.
A two-parameter generalization of differential entropy (1.3) is defined and studied by Varma in 1966. Which is defined as

$$H_{\gamma}^{\beta}(f) = (\beta-\gamma)^{-1} \ln \left(\int_0^{\infty} \frac{1}{f^{1-\gamma-\beta}(x)} \, dx \right); \quad \beta \neq \gamma, \quad \beta - 1 < \gamma < \beta, \quad \beta \geq 1, \tag{1.4}$$

where β, γ are parameters and $f(x)$ is *pdf*.
When $\beta = 1$ result by Varma (1966) redsces to Renyi (1961). A well known one parametric order of generalized measure named Tsallis (1988) is introdsce by Havrda and Charvat in (1967) and was defined as

$$H_{\gamma}(X) = (1-\gamma)^{-1} \int_0^{\infty} f^{\gamma}(x) dx - \frac{1}{1-\gamma} \tag{1.5}$$

where $\gamma \neq 1$, $\gamma > 1$. The non-additive measure named Tsallis has applications in many fields like biological and chemical phenomena.
Data is more valuable with its analytic results that help with interpretation and decision-making. Record statistics study data with more clearance. Record statistics has applications in making policies, justifying data and graphical presentation studies more detail we refer to Arnold et al. (1998) Ahsanullah (2004), and Zahedi and Shakil (2006). Record statistics include both upper and lower values. For a given independent and identically distributed (iid) random variable's series $< X_i >$ have X_l be an upper record value or lower record value if X_l be largest or lowest value compared to all its previous values. The pdf for l^{th} upper record statistics defined as

$$f_l^u(x) = \frac{\{-\ln(\bar{F}(x))\}^{l-1}}{\Gamma(l)} f(x), \; -\infty < x < +\infty. \tag{1.6}$$

If we replace or put value $l = 1$ then equ. (1.6) reduce the parent random variable.
Ahmadi and Fashandi (2008) studied record statistics based on Shannon's entropy measure, the measure for l^{th} upper record defined as

$$H(f_l^u) = -\int_0^{\infty} f_l^u(x) \ln f_l^u(x) dx \tag{1.7}$$

Equation (1.7) can be written as

$$H(f_l^u) = -\int_0^{\infty} \frac{\{-\ln(\bar{F}(x))\}^{l-1}}{\Gamma(l)} f(x) \ln \frac{\{-\ln(\bar{F}(x))\}^{l-1}}{\Gamma(l)} f(x) dx.$$

In 2008 Baratpour et al. proposed and study of characterization properties based on Renyi entropy of order statistics and record values. A two-parametric generalized entropy for record value refers to Kumar (2015). The proposed measure was defined as

$$H_{\gamma}^{\beta}(f_l^u) = (\beta-\gamma)^{-1}\ln\left(\int_0^{\infty}\left(\frac{\{-\ln(\bar{F}(x))\}^{l-1}}{\Gamma(l)}f(x)\right)^{\gamma+\beta-1}dx\right);\quad \beta\neq\gamma,\quad \beta-1<\gamma<\beta,\quad \beta\geq 1, \tag{1.8}$$

A study of Non- additive Entropy measure for record statistics and characterization results and some specific distribution refers to Kumar and Taneja (2015). The proposed measure was defined as

$$H_{\gamma}(f_l^u(x)) = \frac{\int_0^{\infty}\left(\frac{\{-\ln(\bar{F}(x))\}^{l-1}}{\Gamma(l)}f(x)\right)^{\gamma}dx - 1}{1-\gamma} \tag{1.9}$$

The entropy measure study by Shannon (1.1) is useful only for the complete life of a new function or a function that was not used earlier. Actually, this measure has limitations for the study of dynamic life. Ebarahami in 1966 answered the question, of whether a system or function has survived up a time then what about uncertainty in remaining life by studying the concept of entropy measure for a residual lifetime.

If the random variable $[X-t|X>t]$ is a non-negative random variable, $f(x)$ and $F(x)$ are probability density function and probability distribution function then analogous to (1.1) residual entropy measure for a continuous random variable is given as

$$H(X,t) = -\frac{1}{\bar{F}(x)}\int_0^{\infty} f(x)\ln\frac{f(x)}{\bar{F}(x)}dx, \tag{1.10}$$

Kumar and Taneja (2011) have proposed the DCRE of order α and type β, through the relationship

$$\xi_{\alpha}^{\beta}(X;t) = \frac{1}{\beta-\alpha}\log\left[\int_t^{\infty}\bar{F}_t^{\alpha+\beta-1}(x)dx\right], \tag{1.11}$$

and studied its properties. When $\beta=1$, then (1.11) reduce to , *a dynamic cumulative entropy of order* α The quantile approach is more relevant to define a probability distribution compared to the density approach or we can say it provides an alternate study to the probability distribution function.

If $Q(s)$ denotes the quantile function then corresponding to r.v. X and distribution function $F(x)$, quantile function $Q(w)$ is given as

$$Q(w) = F^{-1}(w) = \inf\{x \mid F(x)\geq w\},\ \ 0\leq w\leq 1. \tag{1.12}$$

Sunoj and Sankaran 2012 introduced a quantile-based Shannon entropy function which is defined as

$$H\xi_u(P) = \int_0^1 \ln q(w)dw, \tag{1.13}$$

where $q(w)$ is quantile density function.
More in detail its properties characterization results we can refer to Sunoj and Sankaran (2012), refers to Sunoj in 2013 investigating the quantile-based entropy function in a past lifetime.

Quantile Rényi's entropy function and its residual version were studied by Nanda et al. in 2014, the quantile Rényi measure of parameter γ is defined as

$$\xi_{\gamma}(X; Q(w)) = (1-\gamma)^{-1} \int_0^1 q(p)^{1-\gamma} dp \tag{1.14}$$

Order statistics and record statistics are fundamentally and mathematically interrelated to each other. Entropy-based on quantile of order (γ, β) in order statistics is defined as

$$H^{(\gamma,\,\beta)}(f_{i:n}) = (\beta-\gamma)^{-1} \ln\left(\int_0^{\infty} (f_{i:n}(x))^{\gamma+\beta-1} dx\right);\ \beta \neq \gamma,\ \beta-1<\gamma<\beta,\ \beta \geq 1\ , \tag{1.15}$$

For more details studies of the measure (1.15) we refer to Kumar (2018), a study in detail for dynamic cumulative past quantile entropy ordering we refer to Wang et al. (2021).
The quantile based *dynamic cumulative residual entropy (DCRE)* is defined by

$$\xi(u) = \xi(X; Q(u) = \frac{\log(1-u)}{(1-u)} \int_u^1 (1-p)q(p)dp - (1-u)^{-1} \int_u^1 \log(1-p)(1-p)q(p)dp. \tag{1.16}$$

When $u \longrightarrow 0$, (1.16) reduces to $\xi = -\int_0^1 (\log(1-p))(1-p)q(p)dp$, a quantile version of CRE. Recently, quantile-based Shannon entropy for record statistics has been studied by Dangi and Kumar refer to Dangi and Kumar (2023). The proposed measure is defined as

$$\mathfrak{H}(U_l) = -\int_0^1 \mathfrak{f}_l(w) \ln\{\mathfrak{f}_l(w)\} Q(w) dw \tag{1.17}$$

where $\mathfrak{f}_l(w)$ is quantile version l^{th} upper record defined as

$$\mathfrak{f}_l(w) = \frac{\{-\ln(1-w)\}^{l-1}}{\Gamma(l) Q(w)}; \quad 0 \leq w \leq 1 \tag{1.18}$$

The quantile version of the Rényi entropy and its dynamic versions for record statistics are referred to by Dangi and Kumar (2024). By keeping in mind the importance of quantile function and record statistics and throughout of the literature, the Quantile Study of Generalized Non-Additive Entropy Measure for Record Value is found to be a gap in research. In this manuscript, section 1 concludes the brief introdsction and literature survey. The proposed measure and its importance are devoted to section 2 and the study of the proposed measure for some liftime distributions are included in section 3 and at the end section 4 concludes this manuscript.

2 Quantile-Based Residual Non-Additive Entropy Measure and Record Value

Nonnegative entropy measures (Harda and Charvat 1988) for record statistics are essential for numerous applications. But this measure is only relevant for a new or unused function. if a function is used for some unit of time, then what about residual life reliability or failure? By using a quantile density function of l^{th} upper record statistics (1.18) and Non- additive Entropy for l^{th} upper record statistics (1.9) in this section we proposed a quantile approach of Non- additive Entropy for record statistics, which is defined as

$$H_\gamma(f_{\mathfrak{z}_l}(w), u) = (1-\gamma)^{-1} \int_u^1 \frac{\{f_{\mathfrak{z}_l}(w)\}^\gamma q(w)}{\bar{G}(X_j(u))^\alpha} dw - \frac{1}{1-\gamma}. \tag{2.1}$$

It can be written as

$$H_\gamma(f_{\mathfrak{z}_l}(w), u) = \frac{1}{(1-\gamma)(\Gamma(l;\log(1-w)))^\gamma} \int_u^1 (-\ln(1-w))^{(l-1)\gamma}(q(w))^{1-\gamma} dw - (1-\gamma)^{-1}. \tag{2.2}$$

If we take $u = 1$ the proposed measure 2.2 will reduce to a quantile version of Non- additive Entropy for record statistics, which is given as

$$H_\gamma(f_{\mathfrak{z}_l}(w), u) = \frac{1}{(1-\gamma)(\Gamma(l))^\gamma} \int_0^1 (-\ln(1-w))^{(l-1)\gamma}(Q(w))^{1-\gamma} dw - (1-\gamma)^{-1}, \tag{2.3}$$

and if we take $\gamma = 1$, the equ. (2.3) will reduce to a result obtained by Dangi and Kumar (2023), Which is given as

$$\lim_{\gamma\to 1} H_\gamma(f_{\mathfrak{z}_l}(w)) = \lim_{\gamma\to 1} \left(\frac{1}{(1-\gamma)(\Gamma(l))^\gamma} \int_0^1 (-\ln(1-w))^{(l-1)\gamma}(Q(w))^{1-\gamma} dw - (1-\gamma)^{-1} \right), \tag{2.4}$$

in (2.4), $\lim_{\gamma\to 1} H_\gamma(f_{\mathfrak{z}_l}(w))$ is an indeterminate form so limit can exist by applying L Hospital's rule, the equ. (2.3) will redsce to

$$H_{\mathfrak{z}}(U_l) = -\int_0^1 f_{\mathfrak{z}_l}(w) \ln\{f_{\mathfrak{z}_l}(w)\} Q(w) dw.$$

Theorem 2.1. $H_\gamma(f_l(w), u) = \frac{(1-\gamma)^{-1}}{(\Gamma(l;\log(1-w)))^\gamma} \frac{\Gamma((l-1)\gamma+1)}{(\Gamma(l))^\gamma} E\left[q(1-e^{-V})\right]$, *where* $V \sim \Gamma((l-1)\gamma + 1)$ *and* E *is the expectation.*

Proof. Using the expressions of quantile-based non-additive entropy measure and record value

$$H_\gamma(f_{\mathfrak{z}_l}(w), u) = (1-\gamma)^{-1} \int_u^1 \frac{\{f_{\mathfrak{z}_l}(w)\}^\gamma q(w)}{\bar{G}(X_j(u))^\alpha} dw - \frac{1}{1-\gamma}$$

and

$$H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w), u) = \frac{1}{(1-\gamma)(\Gamma(l;\log(1-w)))^\gamma}\int_u^1 (-\ln(1-w))^{(l-1)\gamma}(q(w))^{1-\gamma}dw - (1-\gamma)^{-1}.$$

putting $-\ln(1-w) = t$ in the above expression then
$dw = e^{-t}dt, \forall\ \ 0 < t < \infty.$
It will converted to

$$(1-\gamma)H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w), u) = \frac{1}{(1-\gamma)(\Gamma(l;\log(1-w)))^\gamma}\int_0^\infty (t)^{(l-1)\gamma}(q(1-e^{-t}))^{1-\gamma}e^{-t}dt - 1, \quad (2.5)$$

it can be written as

$$(1-\gamma)H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w), u) + 1 = \frac{1}{(1-\gamma)(\Gamma(l;\log(1-w)))^\gamma}\int_0^\infty (t)^{(l-1)\gamma}(q(1-e^{-t}))^{1-\gamma}e^{-t}dt,$$

then multiplying and divide by $\Gamma((l-1)\gamma+1)$ to right hand side of the equation, we obtain

$$H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w)) + (1-\gamma)^{-1} = \frac{1}{(1-\gamma)(\Gamma(l;\log(1-w)))^\gamma}(\Gamma(l))^\gamma\int_0^\infty \frac{1}{\Gamma((l-1)\gamma+1)}(t)^{(l-1)\gamma}(q(1-e^{-t}))^{1-\gamma}e^{-t}dt.$$

Since $\frac{1}{\Gamma((l-1)\gamma+1)}(t)^{(l-1)\gamma}(q(1-e^{-t}))^{1-\gamma}e^{-t}$ is the $E\left[q(1-e^{-V})\right]$, then we obtain

$$H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w), u) = \frac{(1-\gamma)^{-1}}{(\Gamma(l;\log(1-w)))^\gamma}\frac{\Gamma((l-1)\gamma+1)}{(\Gamma(l))^\gamma}E\left[q(1-e^{-V})\right].$$

where $V \sim \Gamma((l-1)\gamma+1)$ and E is the expectation. Hence it concludes the results. □

As we know hazard rate function represents the rate of failure in the next small dsration of time when a utility has survived up to a time. The quantile version of hazard rate for l^{th} upper record statistics is defined as

$$H_l(w) = \frac{(-\ln(1-s))^{l-1}}{\Gamma(l;-\ln(1-w))Q(w)}; \quad 0 \le w \le 1, \quad (2.6)$$

where $\Gamma(l;-\ln(1-w))$ is incomplete gamma function.

$$H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w), u) = \frac{(1-\gamma)}{(\Gamma(l;\log(1-w)))}\int_0^1 \{\mathfrak{f}_{\mathfrak{z}_l}(w)\}^\gamma Q(w)dw - 1,$$

it can expressed as

$$Q(s) = \frac{(-\ln(1-w))^{l-1}}{\Gamma(l;-\ln(1-w))H_l(w)}; \quad 0 \le w \le 1,$$

The above expression defined the relation between the quantile density function and quantile version of the hazard rate for l^{th} upper record statistics.

By putting the value of the quantile version of the hazard rate for l^{th} upper record statistics (2.6) in the expression (2.1), we obtained

$$H_\gamma(\mathfrak{f}_l(w),u)=\frac{(-\ln(1-w))^{l-1}}{\Gamma(l;-\ln(1-w))}\int_0^1\{\mathfrak{f}_l(w)\}^\gamma\left(\frac{(-\ln(1-w))^{l-1}}{\Gamma(l;-\ln(1-w))H_l(w)}\right)dw-1,$$

it can be written as

$$H_\gamma(\mathfrak{f}_l(w),u)=\frac{(-\ln(1-w))^{l-1}}{\Gamma(l;-\ln(1-w))}\int_0^1\left(\frac{(-\ln(1-w))^{l-1}}{\Gamma(l;-\ln(1-w))H_l(w)}\right)\{\mathfrak{f}_l(w)\}^\gamma dw-1. \tag{2.7}$$

The above equation presents the quantile approach of Non- additive Entropy for record statistics in the form of a quantile version of hazard rate for l^{th} upper record statistics.

3 Study the quantiled residual non-additive measure for some specific lifetime distribution

In this section we study the quantiled non-additive measure for some specific lifetime distribution like as Uniform distribution, Exponential distribution, Classical Pareto distribution, Rescaled Beta distribution, and Powered distribution.

3.1 The quantiled residual non-additive measure for Uniform distribution

For given non-negative random variable X over $[a,b]$ and their corresponding probability density function $f(x)$ and distribution function $F(x)$. Then

$$F(x)=\frac{x-a}{b-a}\;and\;f(x)=\frac{1}{b-a}.$$

Since the quantile function and quantile density function of uniform distribution are defined as

$$Q(w)=a+(b-a)w$$

and

$$Q(w)=b-a$$

respectively. Then residual quantiled non-additive measure for the Uniform distribution is defined as

$$\begin{aligned}H_\gamma(\mathfrak{f}_l(w),u)&=\frac{(-\ln(1-w))^{l-1}}{\Gamma(l;-\ln(1-w))(1-\gamma)}\int_u^1(-\ln(1-w))^{(l-1)\gamma}(Q(w))^{1-\gamma}dw-(1-\gamma)^{-1}\\&=\frac{(-\ln(1-w))^{l-1}}{\Gamma(l;-\ln(1-w))(1-\gamma)}\int_0^1(-\ln(1-w))^{(l-1)\gamma}(b-a)^{1-\gamma}dw-(1-\gamma)^{-1}\\&=(b-a)^{1-\gamma}\frac{(-\ln(1-w))^{l-1}}{\Gamma(l;-\ln(1-w))(1-\gamma)}\int_0^1(-\ln(1-w))^{(l-1)\gamma}dw-(1-\gamma)^{-1}.\end{aligned}$$

putting $-\ln(1-w)=t$ in the above expression then
$dw = e^{-t}dt, \forall\ 0<t<\infty$.
It will converted to

$$\begin{aligned}
H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w),u) &= (b-a)^{1-\gamma}\frac{(-\ln(1-w))^{l-1}}{\Gamma(l;-\ln(1-w))(1-\gamma)}\int_0^\infty t^{(l-1)\gamma}e^{-t}dt-(1-\gamma)^{-1}\\
&= \frac{(b-a)^{1-\gamma}}{(1-\gamma)(\Gamma(l))^\gamma}(\Gamma(l-1)\gamma+1)-(1-\gamma)^{-1}\\
&= \frac{(b-a)^{1-\gamma}(\Gamma(l-1)\gamma+1)-\Gamma(l;-\ln(1-w))(1-\gamma)}{(1-\gamma)(\Gamma(l))^\gamma}
\end{aligned}$$

The quantile non-additive measure for the Uniform distribution

$$H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w),u) = (1-\gamma)^{-1}\left(\frac{(b-a)^{1-\gamma}(\Gamma(l-1)\gamma+1)-\Gamma(l;-\ln(1-w))(1-\gamma)}{(\Gamma(l))^\gamma}\right) \tag{3.1}$$

If we substitute $\gamma=1$ in the equation (3.1), obtained result will redsce the quantile-based Non- additive Entropy for record statistics for uniform distribution..

3.2 The quantiled non-additive residual measure for Exponential distribution

For a given non-negative continuous random variable X and their corresponding probability density function $f(x)$ and distribution function $F(x)$ respectively. Then

$$F(x) = 1-e^{-\lambda x}.$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0, \lambda>0 \\ 0 & x\leq 0 \end{cases}$$

and corresponding Quantile versions are

$$Q(w) = -\theta^{-1}\ln(1-w)$$

and

$$Q(w) = \frac{1}{(1-w)\theta}\ \forall\theta>0,\ 0\leq w\leq 1,$$

respectively. Then quantiled residual non-additive measure for the exponential distribution is defined as

$$\begin{aligned}
H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w),u) &= \frac{\Gamma(l;-\ln(1-w))}{(\Gamma(l))^\gamma}\int_0^1(-\ln(1-w))^{(l-1)\gamma}(Q(w))^{1-\gamma}dw-(1-\gamma)^{-1}\\
&= \frac{1}{(1-\gamma)\Gamma(l;-\ln(1-w))}\int_0^1(-\ln(1-w))^{(l-1)\gamma}\left(\frac{1}{(1-w)\theta}\right)^{1-\gamma}dw-(1-\gamma)^{-1}
\end{aligned}$$

putting $-\ln(1-w) = t$ in the above expression then
$(1-w) = e^{-t}\, dw = e^{-t}dt,\ \forall\ \ 0 < t < \infty.$
The quantile non-additive measure for the exponential distribution is defined as

$$H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w)) = (1-\gamma)^{-1}\left(\frac{\Gamma((l-1)\gamma+1)\Gamma(l;-\ln(1-w)) - (\Gamma(l))^\gamma(\gamma)^{(l-1)+1}}{\gamma(\gamma)^{(l-1)+1}}\right) \tag{3.2}$$

If we substitute $gamma = 1$ in the equation (3.2), obtained result will reduce the quantile-based Non- additive Entropy for record statistics for exponential distribution as results by Sunoj and Sankaran (2012).

3.3 The quantiled non-additive measure for Classical Pareto distribution

For given non-negative continuous random variable X and its corresponding probability density function $f(x)$ and distribution function $F(x)$ respectively. Then quantile versions are

$$Q(w) = (1-w)^{-\frac{1}{\theta}}$$

and

$$q(w) = \frac{1}{\theta}(1-w)^{-(\frac{1+\theta}{\theta})},$$

respectively. Then quantile non-additive measure for Classical Pareto distribution is defined as

$$\begin{aligned} H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w)) &= \frac{(1-\gamma)^{-1}}{(\Gamma(l))^\gamma}\int_0^1 (-\ln(1-w))^{(l-1)\gamma}(Q(w))^{1-\gamma}dw - (1-\gamma)^{-1} \\ &= \frac{1}{(1-\gamma)(\Gamma(l))^\gamma}\int_0^1 (-\ln(1-w))^{(l-1)\gamma}\left(\frac{1}{\theta}(1-w)^{-(\frac{1+\theta}{\theta})}\right)^{1-\gamma}dw - (1-\gamma)^{-1} \end{aligned}$$

putting $-\ln(1-w) = t$ in the above expression then
$(1-w) = e^{-t}\, dw = e^{-t}dt,\ \forall\ \ 0 < t < \infty.$
It will converted to

$$H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w)) = \frac{1}{(1-\gamma)(\Gamma(l))^\gamma}\int_0^\infty t^{(l-1)\gamma}\left(\frac{1}{\theta}e^{(\frac{1+\theta}{\theta})t}\right)^{1-\gamma}e^{-t}dt - (1-\gamma)^{-1}$$

It can be written as

$$\begin{aligned} H_\gamma(\mathfrak{f}_{\mathfrak{z}_l}(w)) &= \frac{(1-\gamma)^{-1}\Gamma((l-1)\gamma+1)}{(\Gamma(l))^\gamma\big(\gamma(1-\frac{1}{\theta})\big)^{((l-1)\gamma+1)}} - (1-\gamma)^{-1} \\ &= \frac{(1-\gamma)^{-1}}{(1-w)^{-\frac{1}{\theta}}}\left(\frac{\Gamma((l-1)\gamma+1) - (\Gamma(l))^\gamma\big(\gamma(1-\frac{1}{\theta})\big)^{((l-1)\gamma+1)}}{(\Gamma(l))^\gamma\big(\gamma(1-\frac{1}{\theta})\big)^{((l-1)\gamma+1)}}\right) \end{aligned} \tag{3.3}$$

If we substitute $\gamma = 1$ in the equation (3.3), obtained result will reduce the quantile-based Non- additive Entropy for record statistics for exponential distribution.

3.4 The quantiled residual non-additive measure for Power Function distribution

Then, the quantile function and quantile density function for the Power function are

$$Q(w) = aw^{\frac{1}{b}}$$

and

$$Q(w) = \frac{a}{b}w^{\frac{1}{b}-1}\ \forall a \neq 0\, and\, b \neq 1$$

respectively. The quantile non-additive measure for the Power Function distribution is defined as

$$H_\gamma(\mathfrak{f}_{\mathfrak{Z}_l}(w)) = \frac{1}{(1-aw^{\frac{1}{b}})(1-\gamma)(\Gamma(l))^\gamma}\int_0^1 (-\ln(1-w))^{(l-1)\gamma}\left(\frac{a}{b}w^{\frac{1}{b}-1}\right)^{1-\gamma} dw-(1-\gamma)^{-1} \quad (3.4)$$

3.5 The quantiled Residual non-additive measure for Rescaled Beta distribution

Then Quantile function and the quantile density function for the Rescaled Beta distribution are defined as

$$Q(w) = R\left(1-(1-w)^{\frac{1}{C}}\right)\ C,\ R\ > 0$$

and

$$q(w) = \frac{R}{C}(1-w)^{\frac{1}{C}-1}.$$

Then quantile-based residual Non- additive Entropy measure for record statistics for Rescaled beta distribution is defined as

$$\begin{aligned} H_\gamma(\mathfrak{f}_{\mathfrak{Z}_l}(w), u) &= \frac{1}{(1-\gamma)(\Gamma(l))^\gamma}\int_0^1 (-\ln(1-w))^{(l-1)\gamma}(Q(w))^{1-\gamma}dw-(1-\gamma)^{-1} \\ &= \frac{(1-R\left(1-(1-w)^{\frac{1}{C}}\right))-1}{(1-\gamma)(\Gamma(l;\log(1-w))^\gamma}\int_0^1 (-\ln(1-w))^{(l-1)\gamma}\left(\frac{R}{C}(1-u)^{\frac{1}{C}-1}\right)^{1-\gamma} dw-(1-\gamma)^{-1} \end{aligned}$$

substituting $-\ln(1-s) = t$ in the above expression then
$(1-s) = e^{-t}\, ds = e^{-t}dt,\ \forall\ \ 0 < t < \infty.$
The expression will be converted to

$$\begin{aligned} H_\gamma(\mathfrak{f}_{\mathfrak{Z}_l}(w), u) =& \frac{\left(1-R\left(1-(1-w)^{\frac{1}{C}}\right)\right)^{-1}}{(1-\gamma)(\Gamma(l;\log(1-w))^\gamma}\int_0^\infty t^{(l-1)\gamma}\left(\frac{R}{C}(e)^{-(\frac{1}{C}-1)t}\right)^{1-\gamma} e^{-t}dt-(1-\gamma)^{-1} \\ &= \frac{R^\gamma}{(1-\gamma)(C\Gamma(l))^\gamma}\frac{\Gamma((l-1)\gamma+1)}{\left[(\frac{1}{C}-1)(1-\alpha)+1\right]^{((l-1)\gamma+1)}}-(1-\gamma)^{-1} \end{aligned}$$

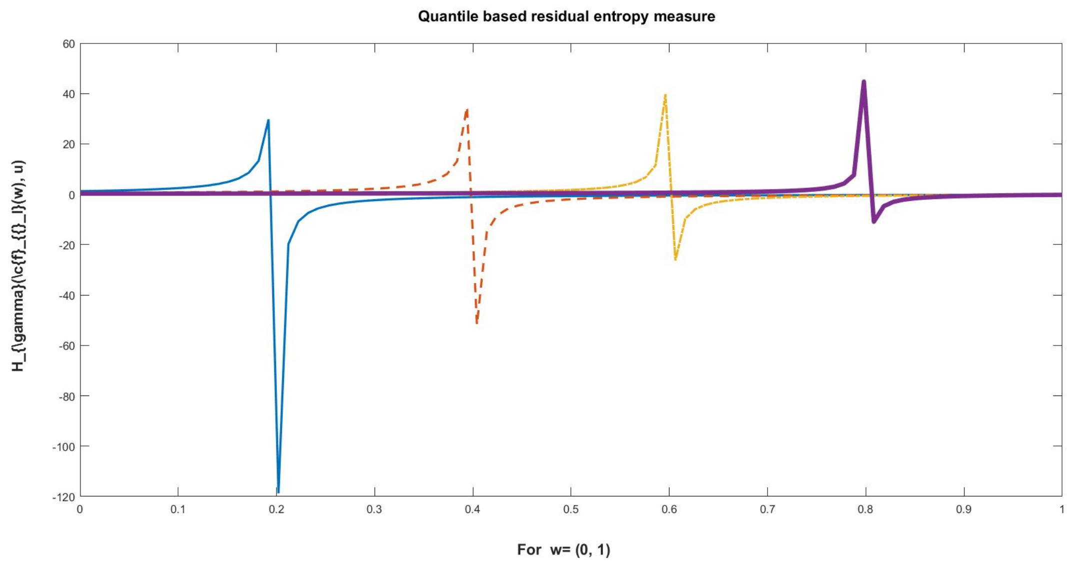

Figure 1: The quantiled residual non-additive measure for Rescaled Beta distribution

$$H_{\gamma}(\mathfrak{f}_{\varsigma_l}(w), u) = \frac{(1 - R\left(1 - (1-w)^{\frac{1}{C}}\right))^{-1}}{(1-\gamma)(\Gamma(l; \log(1-w))^{\gamma}} \left(\frac{R^{\gamma}\Gamma((l-1)\gamma+1) - (C\Gamma(l))^{\gamma}\left[(\frac{1}{C}-1)(1-\alpha)+1\right]^{((l-1)\gamma+1)}}{(C\Gamma(l))^{\gamma}\left[(\frac{1}{C}-1)(1-\alpha)+1\right]^{((l-1)\gamma+1)}} \right) \tag{3.5}$$

If we substitute $\gamma = 1$ in the equation (3.5), obtained result will redsce the quantile-based Non- additive Entropy for record statistics for exponential distribution..

4 Conclusion

If a function is used for some unit of time, then what about residual life reliability or failure? In this research paper, we study the quantile approach of Residual Non- additive Entropy for record statistics and some specific distributions. For numerous applications, non-negative entropy measures (Harda and Charvat 1988) for record statistics are essential. Also, we study the proposed measure in the hazard rate function. Expressing the proposed measure in terms of expectation provides an alternate approach to solving the measure for specific distributions.

5 Conflict of Interest

The corresponding authors assure that there is no conflict of interest on behalf of all authors.

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